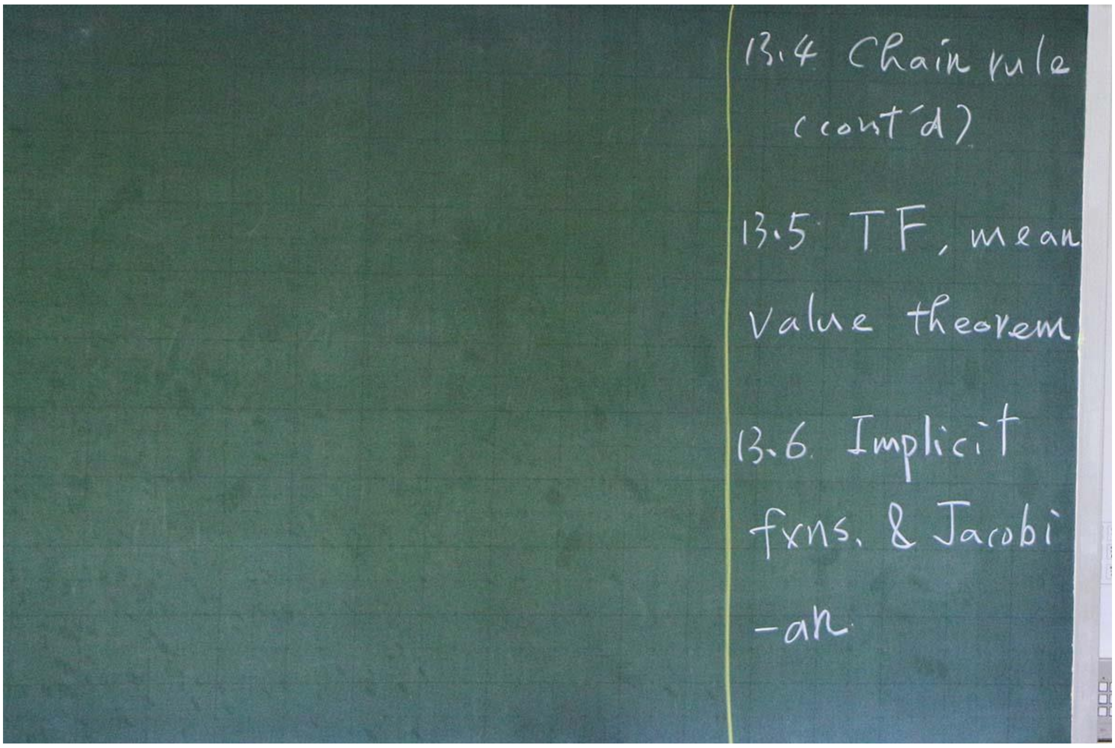

13.4 Chain rule
(cont'd)
13.5 TF, mean
value theorem
13.6 Implicit
fxns. & Jacobi
-an


13.4 Chain rule
(cont'd)
13.5 TF, mean
theorem
Implicit
Jacobi

<Ex.2>

Let $f(u,v) = uv^2$, where $u = 3x - y$, $v = x^2 y$

$f(u,v) = f(u(x,y), v(x,y)) \equiv F(x,y)$.

composite fxn.

Find $\partial F/\partial x$

Sol.

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$= v^2 \cdot 3 + 2uv \cdot 2xy$$

$$= 15x^4y^2 - 4x^3y^3 \quad \#$$

<comments, x2>

1. $\partial F/\partial x$ holds (($\because$ y fixed)

2. $\dfrac{\partial F(x,y)}{\partial x}$, $\dfrac{\partial f(u,v,x,y)}{\partial x}$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\underline{\frac{\partial u}{\partial x}} + \frac{\partial f}{\partial v}\underline{\frac{\partial v}{\partial x}} + \frac{\partial f}{\partial x}\cancel{\frac{\partial x}{\partial x}}^{\nearrow 1} + \frac{\partial f}{\partial y}\cancel{\frac{\partial y}{\partial x}}^{\nearrow 0}$$

$u(\underline{x}, \underset{x}{\underline{y}})$

$$= f_u u_x + f_v v_x + f_x \quad \#$$

f_x

$$\cancel{\frac{\partial f}{\partial x}} = f_u u_x + f_v v_x + \cancel{f_x} \Longrightarrow f_u u_x + f_v v_x = \cancel{\ }$$

13.5 Taylor's formula (TF), Taylor's series (T.S $f|_a$), mean value theorem

I Single variable

$$\int_a^x f'(x)\,dx = f(x) - f(a) \qquad \text{"fundamental theorem"}$$

$$\underline{f(x)} = f(a) + \underline{\int_a^x f'(x)\,dx} \quad \text{———} \quad (1)$$

Similarly, $f'(x) = f'(a) + \int_a^x f''(x)\,dx$ ——— (2)

Casting (2) into (1) gives,

$$f(x) = f(a) + \int_a^x \left[f'(a) + \int_a^x f''(x)\,dx \right] dx$$

$$= f(a) + f'(a)(x-a) + \int_a^x \int_a^x f''(x)\,dx\,dx \qquad \text{—— (3)}$$

Next, by repeating this process, we obtain,

$$f(x) = f(a)\cdot \underset{(x-a)^0}{1} + \frac{f'(a)}{1!}\underline{(x-a)} + \frac{f''(a)}{2!}\underline{(x-a)^2} + \cdots$$

$$+ \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \underbrace{\int_a^x \cdots \int_a^x}_{n} f^{(n)}(x)\,(dx)^n \qquad \text{—— (4)}$$

$$\searrow R_n(x)$$

(4) is called TF w/ the remainder

$R_n(x)$

Suppose that $f^{(n)}(x)$ on $[a,x]$ has a minimal value, m, maximal value, M.

$$\int_a^x \cdots \int_a^x \underline{\underline{m}}\,(dx)^n \le R_n(x) \le \int_a^x \cdots \int_a^x \underline{\underline{M}}\,(dx)^n$$

$$\frac{m}{n!}(x-a)^n \le R_n(x) \le \frac{M}{n!}(x-a)^n \rightarrow \text{polynomial fxns!}$$

(4) turns to be,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$$

$$+ \frac{f^{(n)}(\xi)}{n!}(x-a)^n \quad \#$$

$$TF \overset{?}{=} TS$$

$TSf\big|_{x=a} \rightarrow$

1. Let $n \rightarrow \infty$
2. Drop $R_n(x)$!

If TS converges, and converges to $f(x)$, then,

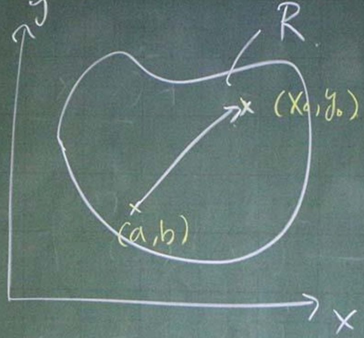
$$f(x) = TS\, f\Big|_{x=a} = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \#$$

I. 2. Mean Value theorem $(n=1)$

$$f(x) = f(a) + f'(\xi)(x-a) \qquad \#$$

II Multi-variable cases

$f(x,y)$.



$$\begin{cases} x = a + (x_0 - a)t \\ y = b + (y_0 - b)t \end{cases} ; \quad 0 \le t \le 1$$

$$f(x(t), y(t)) \equiv F(t).$$

$$F(t) = F(0) + \frac{F'(0)}{1!}t + \frac{F''(0)}{2!}t^2 + \cdots$$

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x_0-a) + f_y(y_0-b)$$

$$= D F \quad , \text{ where } D \equiv \left[(x_0-a)\frac{\partial}{\partial x} + (y_0-b)\frac{\partial}{\partial y}\right]$$

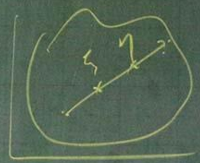
$$F''(t) = D^2 F = \left[(x_0-a)\frac{\partial}{\partial x} + (y_0-b)\frac{\partial}{\partial y}\right]$$

$$\cdot\left[(x_0-a)f_x + (y_0-b)f_y\right] = (x_0-a)^2 f_{xx} + (x_0-a)$$

$$\times (y_0-b) f_{xy} + (x_0-a)(y_0-b) f_{yx} + (y_0-b)^2 f_{yy}$$

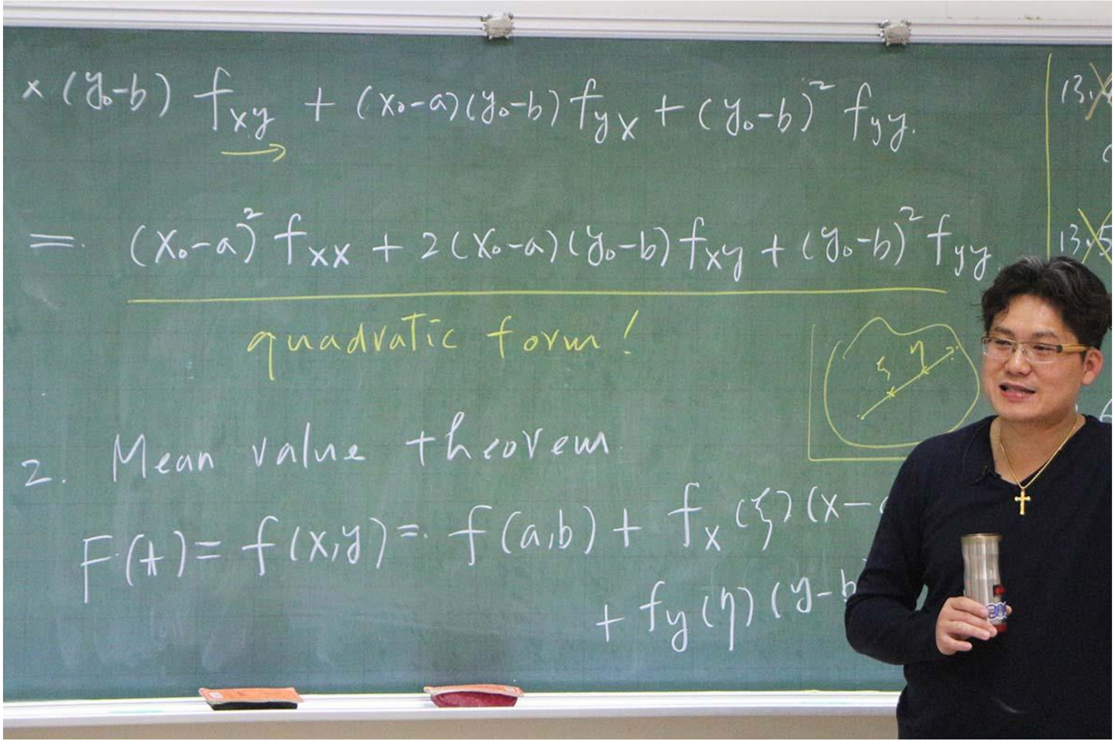
$$= (x_0-a)^2 f_{xx} + 2(x_0-a)(y_0-b) f_{xy} + (y_0-b)^2 f_{yy}$$

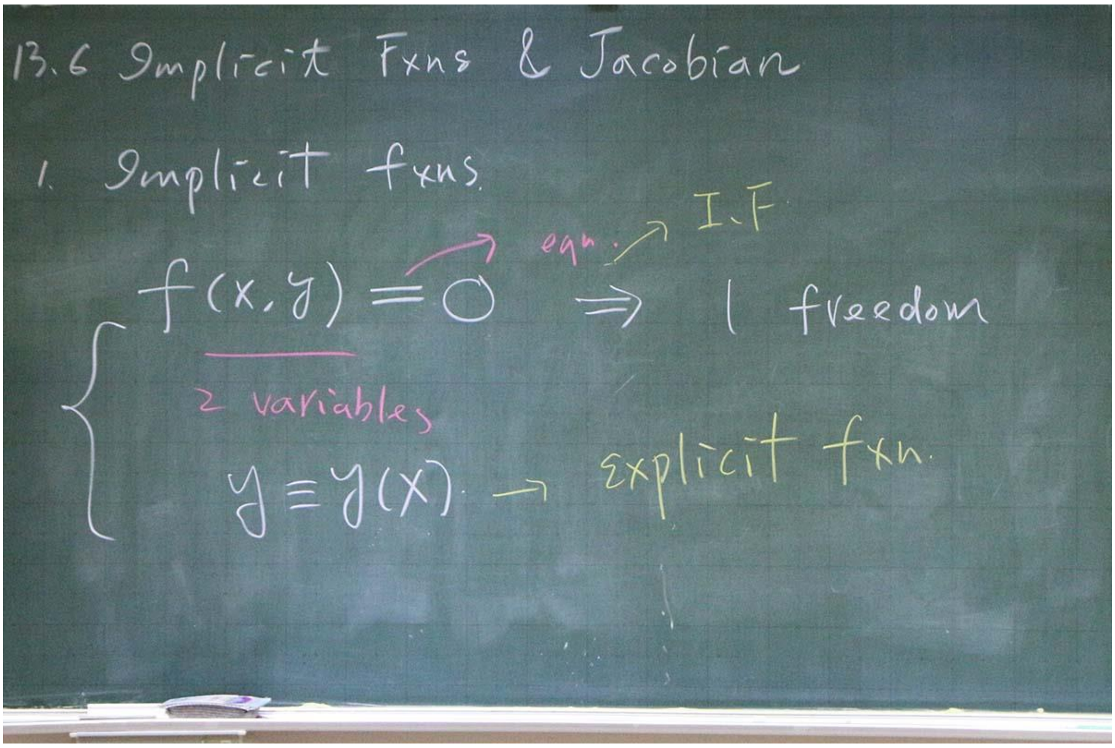
quadratic form!



2. Mean value theorem

$$F(t) = f(x,y) = f(a,b) + f_x(\xi)(x-a) + f_y(\eta)(y-b)$$


× (y₀-b) f_xy + (x₀-a)(y₀-b) f_yx + (y₀-b)² f_yy
= (x₀-a)² f_xx + 2(x₀-a)(y₀-b) f_xy + (y₀-b)² f_yy
quadratic form!
2. Mean value theorem
F(t) = f(x,y) = f(a,b) + f_x(ξ)(x-
+ f_y(η)(y-b


13.6 Implicit Fxns & Jacobian
1. Implicit fxns
eqn
I.F
f(x, y) = 0 ⇒ 1 freedom
2 variables
y ≡ y(x) → explicit fxn

< Implicit fxn. theorem >

$y(x)$ exists uniquely, $\dfrac{\partial f(x,y)}{\partial y} \neq 0$

$$y(x) = y(a) + \frac{y'(a)}{1!}(x-a) + \cdots$$

e.g., $$\frac{\partial f}{\partial X} = \frac{\partial f(x,y)}{\partial X} = \frac{\partial f}{\partial X}\frac{dX}{dX} + \frac{\partial f}{\partial y}\frac{dy}{dX}$$

$$= f_x + f_y y' = 0$$

$$\therefore \quad y' = \frac{dy}{dX} = \frac{-f_x}{f_y} \qquad \text{———} (1)$$

$$y'' = \frac{d(y')}{dX} = \frac{d}{dX}\left(-\frac{f_x}{f_y}\right) = \frac{-\left[f_y \dfrac{df_x}{dX} - f_x \dfrac{df_y}{dX}\right]}{(f_y)^2} \qquad \text{———} (2)$$

where, $$\frac{df_x}{dX} = \frac{\partial f_x}{\partial x}\frac{dX}{dX} + \frac{\partial f_x}{\partial y}\frac{dy}{dX}$$

$$= f_{xx} + f_{xy} y' \qquad \text{———} (3a)$$

$$\frac{df_y}{dx} = \frac{\partial f_y}{\partial x}\frac{dx}{dx} + \frac{\partial f_y}{\partial y}\frac{dy}{dx} = f_{yx} + f_{yy}\, y'$$

2.

$$\begin{cases} f(\underline{x,y}\,;\,\underline{u,v}) = 0 \\ g(x,y\,;\,u,v) = 0 \end{cases}$$

2 sets , 2 variables

$\longrightarrow$ implicit

$u \equiv u(x,y)\,;\; v(x,y) \rightarrow$ explicit

The T.S of u & v are,

$$u(x,y) = u(x_0,y_0) + \underline{u_x}\,(x-x_0) + \underline{u_y}\,(y-y_0) + \cdots$$

$$v(x,y) = v(x_0,y_0) + \underline{v_x}\,(x-x_0) + \underline{v_y}\,(y-y_0) + \cdots$$

$$
\begin{cases}
f(x,y,u(x,y),v(x,y)) \equiv F(x,y) \\
g(x,y,u(x,y),v(x,y)) \equiv G(x,y)
\end{cases}
\qquad \text{———— } 1(a,b)
$$

$$
\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x}\cancelto{1}{\frac{\partial x}{\partial x}} + \frac{\partial f}{\partial y}\cancelto{0}{\frac{\partial y}{\partial x}} + \underbrace{\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}}_{f_u u_x} + \underbrace{\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}}_{f_v v_x} = 0
$$

f_y

$$
f_u u_x + f_v v_x = -f_x \qquad \text{———— } 2(a)
$$

$$
\frac{\partial G}{\partial x} = \underbrace{\frac{\partial g}{\partial x}}_{g_x}\cancelto{1}{\frac{\partial x}{\partial x}} + \frac{\partial g}{\partial y}\cancelto{0}{\frac{\partial y}{\partial x}} + \underbrace{\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}}_{g_u u_x} + \underbrace{\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}}_{g_v v_x} = 0
$$

$$
g_u u_x + g_v v_x = -g_x \qquad \text{———— } 2(b)
$$

$$
\frac{\partial F}{\partial y} = \frac{\partial f}{\partial x}\cancelto{0}{\frac{\partial x}{\partial y}} + \underbrace{\frac{\partial f}{\partial y}}_{f_y}\cancelto{1}{\frac{\partial y}{\partial y}} + \underbrace{\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}}_{f_u u_y} + \underbrace{\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}}_{f_v v_y}
$$

$$f_u u_y + f_v v_y = -f_y$$

Likewise, you can compute $\dfrac{\partial G}{\partial X}$.

Now, let's compare 2 (a, b)

$$\begin{cases} f_u \underline{u_x} + f_v \underline{v_x} = -f_x \\ g_u \underline{u_x} + g_v \underline{v_x} = -g_x \end{cases} \quad \text{——— } 2\,(a, b)$$

PS

$$AX = C \quad ; \quad \begin{bmatrix} & \\ & \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$x_j = \frac{\sum A_{ji} c_i}{\det A} \rightarrow \text{co-factor expansion}$$

"Cramer's rule"

for example,

$$u_x = \frac{\overbrace{\begin{vmatrix} -f_x & f_v \\ -g_x & g_v \end{vmatrix}}^{\to\, -J(x,v)}}{\underbrace{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}}_{\det A\ \equiv\ J(u,v)}} \; ; \quad v_x = \frac{\overbrace{\begin{vmatrix} f_u & -f_x \\ g_u & -g_x \end{vmatrix}}^{-J(u,x)}}{\underbrace{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}}_{\det A\ \equiv\ J(u,v)}}$$

< I.F theorem for multivariable cases >

$$\begin{cases} f_1(\underline{x_1, \cdots, x_n}\,;\ \underline{u_1, \cdots, u_n}) = 0 \\ \quad\vdots \\ f_n(x_1, \cdots, x_n\,;\ u_1, \cdots, u_n) = 0 \end{cases}$$

$\begin{cases} 2 \text{ sets} \\ 1 \text{ set} \to n \text{ variables} \end{cases}$

$u_1(x_1, \cdots, x_n)$, $u_2(x_1, \cdots, x_n)$

...

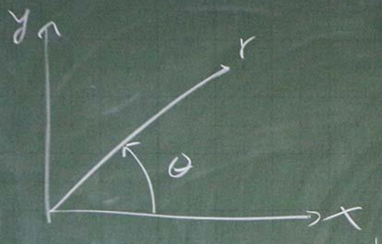
$$\begin{vmatrix} \dfrac{\partial f_1}{\partial u_1} & \cdots & \dfrac{\partial f_1}{\partial u_n} \\ \vdots & & \\ \dfrac{\partial f_n}{\partial u_n} & \cdots & \dfrac{\partial f_n}{\partial u_n} \end{vmatrix} = \frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(u_1, \cdots, u_n)}$$

$$= J(u_1, u_2, \cdots, u_n) \neq 0 \;!!$$

Comment:

Jacobian <u>not only</u> identifies the unique existence of I.F., <u>but also</u> helps calculate partial derivatives!

<Ex.1> Polar coordinates.



$$f = x - r\cos\theta = 0$$

$$g = y - r\sin\theta = 0$$

$$T(x,y) \to T(x(r,\theta), y(r,\theta))$$

$x = r\cos\theta$

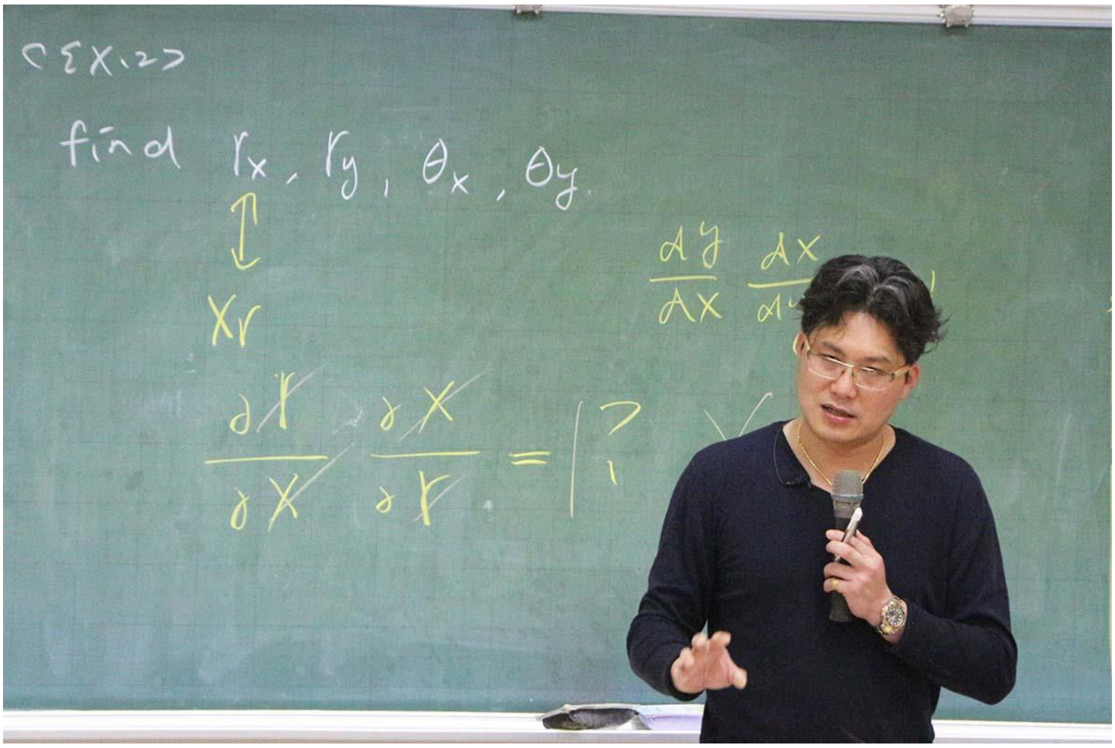
$y = r\sin\theta$

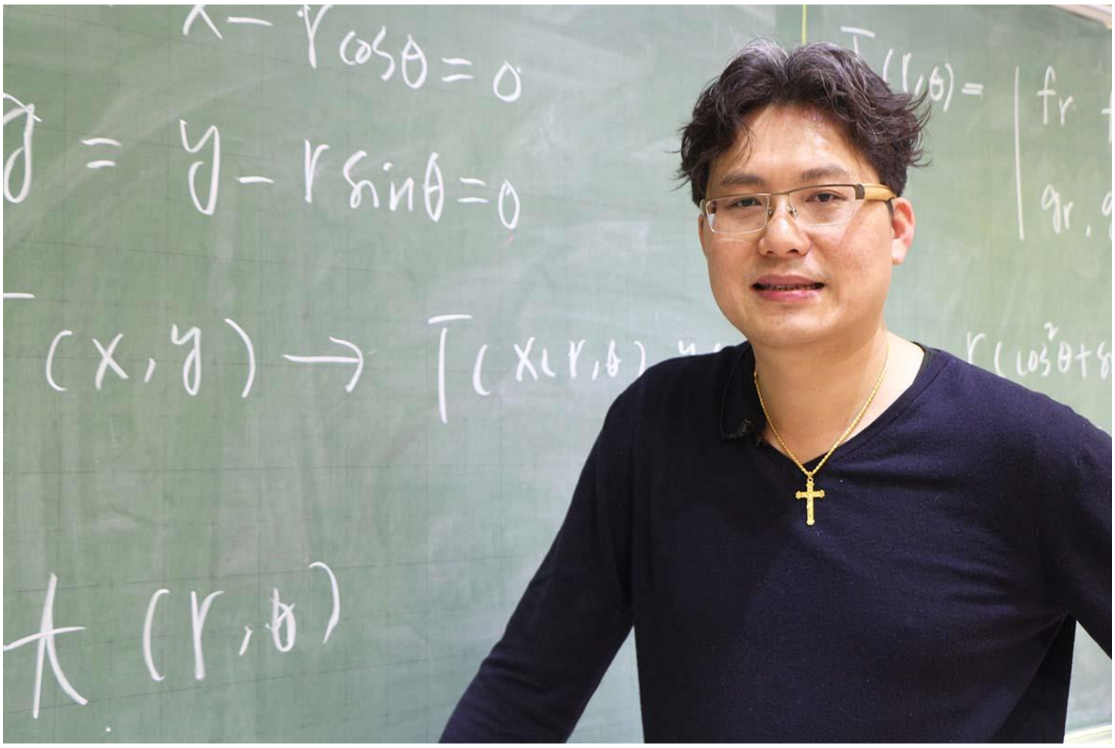
$$= \bar{T}(r,\theta)$$

$$J(r,\theta) = \begin{vmatrix} f_r & f_\theta \\ g_r, & g_\theta \end{vmatrix} = \begin{vmatrix} -\cos\theta & +r\sin\theta \\ -\sin\theta & -r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r \neq 0$$

2 $\quad \dfrac{\partial T}{\partial x}, \dfrac{\partial T}{\partial y} \to \dfrac{\partial \bar{T}}{\partial r}, \dfrac{\partial \bar{T}}{\partial \theta}$


<EX.2>
find rx, ry, θx, θy
Xr


y - r sinθ = 0